AUT.360 Distributed Control and Optimization of Cyber-Physical Systems
-Midterm check-
Date: 20.2.2023

Note:

1. **Write your name and student ID on every answer sheet.**
2. This is an individual test and thus no collaboration is allowed.
3. Write your answers clearly.

Problem 1 (40 points). Consider the graph $\mathcal{G}$ depicted in Figure 1 and answer the following questions.

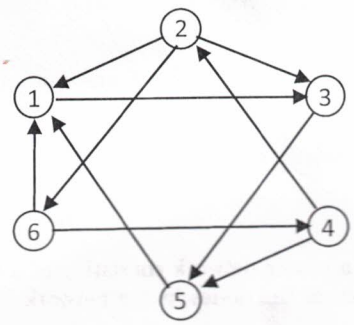

Figure 1: A graph $\mathcal{G}$

(i) Find all the directed paths, if any, from node 2 to node 5 and from node 1 to node 6. (5 points)

(ii) Is the graph strongly connected? Explain your answer. (5 points)

(iii) Write the Laplacian matrix L corresponding to the graph. (5 points)

(iv) By analyzing the network's structure, write the sparsity structure of the left eigenvector (which ones of its entries are zero and non-zero) corresponding to $\lambda_1(L) = 0$. Explain your answer.
Example: the following sparsity structure of the vector

$$q_1 = \begin{bmatrix} * \\ * \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

means that its first and second entries are non-zero while the other entries are zero. (10 points)
remark: note that a null vector cannot be an eigenvector.

(v) Let $x_i(t) \in \mathbb{R}$ denotes the state of node i. Assume that all the nodes execute a consensus algorithm whose dynamics in a compact form is written as $\dot{x} = -Lx$ where $x = [x_1, \cdots, x_6]^T$. Will the state of all the nodes reach a consensus? if the state of all the nodes reach a consensus, please write the consensus value as a function of the initial states of the nodes. Explain your answer. (15 points)

Problem 2 (15 points). For each matrix given in the following subproblems, please answer (and give your reason) whether the matrix is semi-covergent (and not convergent) or not semi-convergent or it cannot be decided based on the provided information.

1. Matrix $P_1 \in \mathbb{R}^{4\times 4}$ whose eigenvalues are given by $\lambda_j(P_1) = \{1, 0.6, \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\}$. (5 points)
2. Matrix $P_2 \in \mathbb{R}^{5\times 5}$ whose eigenvalues are given by $\lambda_j(P_2) = \{1, 1, 0.9, 0.6, 0.4\}$. (5 points)
3. Matrix $P_3 \in \mathbb{R}^{3\times 3}$ whose Jordan form is given by

$$J(\Lambda) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \text{ (5 points)}$$

Problem 3 (10 points). Please answer whether the following statements are true or false and give your reason.

1. If the Laplacian matrix of a graph has exactly one eigenvalue equals to zero, then the associated graph is strongly connected. (5 points)
2. The eigenvalues of a Laplacian matrix corresponding to a strongly connected and balanced graph are always real. (5 points)

Problem 4 (35 points). Consider a sensor network consisting of 5 nodes where the state of each node is denoted by $x_i(k) \in \mathbb{R}$. The dynamics of the nodes in the network is given by the following discrete-time consensus

$$x(k+1) = Px(k)$$

where matrix $P = I_n - \epsilon L$, vector $x(k) = [x_1, \cdots, x_5]^T$, scalar $\epsilon > 0$ and L represents the Laplacian matrix associated with the communication network topology. Assume that matrix P is given by

$$P = \begin{bmatrix} 0.8 & 0 & 0.2 & 0 & 0 \\ 0.2 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0.2 \\ 0 & 0 & 0.2 & 0.6 & 0.2 \\ 0 & 0.2 & 0 & 0.2 & 0.6 \end{bmatrix}. \tag{1}$$

Answer the following questions.

(i) Draw the comunication network topology associated with matrix P in (1). (5 points)

(ii) Show that the states of all the nodes under the above discrete-time dynamics reach a consensus (note: it is not required to explicitly calculate the left eigenvectors). (15 points)

(iii) Is the consensus value equal to the average of the initial states of all the nodes for any $x(0)$? if not, is it possible to achieve average consensus for any $x(0)$ by adding one link to the network (also write where to add the link and the new matrix P which ensures the average consensus)? Explain your answer. (15 points)