Please answer the following four questions, on a separate paper. No books or calculators.
A Collection of Formulas will be handed. Kaavakokoelma jaetaan.

1. Let $\mathcal{P} = \{p, q, r\}$ be a set of propositions. For each of the following, write a propositional formula over $\mathcal{P}$ in which *all* the propositional symbols appear in the formula, and the formula has the required property.

   (a) a formula ϕ_1 that is a tautology

   (b) a formula ϕ_2 that is a contradiction

   (c) a formula ϕ_3 that is neither a contradiction nor a tautology

2. Let R be a binary relation over $\mathbb{N}$ that is defined recursively as follows:

   1. $\forall x \in \mathbb{N} : \langle x, x \rangle \in R$
   2. If $\langle x, y \rangle \in R$ and $\exists k, n, m \in \mathbb{N} : x = kn, y = km$, then $\langle x, m \rangle \in R$
   3. Nothing else is in R.

   Answer the following questions about R:

   (a) Is R a partial order? Prove or show a counterexample.

   (b) Is it true that if $\langle x, y \rangle$ and $\langle z, y \rangle$, then $\langle x + z, y \rangle$? Prove or disprove.

   (c) Is it true that if $\langle x, y \rangle$ and $\langle z, y \rangle$, then $\langle xz, y \rangle$? Prove or disprove.

3. Prove the following claims:

   (a) For arbitrary $a, b \in \mathbb{R}$, $n^2 + an + b \in \Theta(n^2)$ (Hint[1])

   (b) There exists a function $f(n)$ such that $f(n) \in \Omega(n)$ and $f(n) \in O(n^2)$, but $f(n) \notin \Theta(n^a)$ for any fixed a. (Hint[2])

   (c) If $\epsilon > 0$, then $n \cdot \log_2 n \in O(n^{1+\epsilon})$ (Hint[3])

4. Use syllogisms and logical equivalences to prove the following. In b) define the appropriate predicates and formulate the premises in predicate logic first.

   (a) $\forall x : (P(x) \to (Q(x) \vee R(x))) \wedge \exists y : (P(y) \wedge \neg R(y)) \Rightarrow \exists z : Q(z)$

   (b) All ravens are black. Some crows are black. All black birds are either crows or ravens. Pete is black. Pete is not a raven and Pete is not a crow. Therefore, Pete is not a bird.

[1] a and b are still constants, they don't change when n does, so even if c depends on them, it is still a constant as well.

[2] This was actually mentioned during the lectures

[3] There is a formula in the collection that you can use.

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