MATH.MA.210 Discrete mathematics Exam, 08.05.2023

Answer **at least four and at most five** questions out of the following six. If you answer 5 questions, your points total will be counted. If you answer 4 questions, the points will be multiplied by 1.25. Indicate on the first page how many you are submitting. All questions have the same value.

Questions

1. We define the following propositions:
   p means "It is raining", q means "It is cold", and r means "It is windy".
   Also we define the following predicates:
   $A(x)$ means person x is wearing a jacket. $B(x)$ means the person x is wearing waterproof boots. $C(x)$ means x goes out.
   The universal set is some set of people, that includes at least the person "Jack".

   (a) Write the following sentence as a formula: "If Jack goes out and if it is raining, then Jack wears waterproof boots."

   (b) Write the following statement as a formula "It is raining and it is cold and it is windy". From now on consider this formula as the definition of "The weather is bad".

   (c) Write the formula "If the weather is bad, no one goes outside".

2. Prove the following is true for $n \in \mathbb{Z}_+$ by induction:

$$\left(\sum_{i=1}^{n}(2i-1)\right) = n^2$$

3. The relation $\sim$ is defined on $\mathbb{Z}$ as follows: $x \sim y$ if and only if

$$x \equiv y \bmod 6$$

   Prove that $\sim$ is an equivalence relation. Describe the equivalence classes verbally and give examples of an element in each equivalence class.

4. Let A and B be sets such that $|A| = m$ and $|B| = n$, and assume that $m \leq n$.

   (a) How many different functions $f : A \to B$ exist?

   (b) How many of these functions are injections?

   (c) If $m = n$, prove that an injection must be a bijection.

5. Let the operation $\star$ be defined for *odd integers* $\mathbb{Z}^{odd}$ such that

$$x \star y = x + y + 1$$

   Prove that $\mathbb{Z}^{odd}$ with the operation $\star$ forms a group, and explain what is the identity element of the group, and how to calculate the inverse for each element.

6. (a) What is the remainder when 4^{119} is divided by 5?

   (b) Which of the following equations have unique solutions, and why/why not? List all solutions (i.e., all equivalence classes, obviously, as there are infinite many solutions when a solution exists)

      i. $4x \equiv 7 \bmod 12$

      ii. $4x \equiv 7 \bmod 11$

      iii. $2x \equiv 6 \bmod 12$

APPENDIX: Some helpful formulas and definitions

Definition 1. If a relation is reflexive, symmetric, and transitive, we call it an *equivalence relation* or simply an *equivalence.*

Definition 2. A *group* is a pair $(G, \bullet)$ where G is a set $\bullet$ is an operation on G with the following properties

1. Associativity: $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ for a, b, and $c \in G$.

2. Identity element: There exists an element $e \in G$ such that $e \bullet a = a \bullet e = a$ for every $a \in G$.

3. Inverse: for every $a \in G$ there is an element $a^{-1} \in G$ such that $a \bullet a^{-1} = a^{-1} \bullet a = e$.

Definition 3. Let $n \in \mathbb{Z}_+$. If for two numbers $a, b \in \mathbb{Z}$ we have $n \mid (a - b)$, we say that a is *congruent* with b *modulo* n and we denote $a \equiv b \pmod{n}$ or $a \equiv_n b$.