MATH.MA.210 Discrete mathematics Exam, 08.05.2024

Answer **five** questions out of the following six. All questions have the same value.

Questions

1. We define the following propositions:
   p means "It is lunchtime", q means "It is a working day", and r means "It is sunny".
   Also we define the following predicates:
   $A(x)$ means person x is hungry. $B(x)$ means the person x eats at the university restaurant.
   $C(x)$ means x goes out.
   The universal set is some set of people, that includes at least the person "Jack".

   (a) Write the following sentence as a formula: "If it is sunny, then Jack will go out, and if he is out and hungry, he will eat at the university restaurant"

   (b) Write the following sentence as a formula: "If it is not sunny, no one will go out on a weekday"

   (c) Write the following sentence as a formula: "No one eats at the university restaurant if it is not lunchtime and a working day".

2. Prove the following is true for $n \in \mathbb{Z}_+$ using induction:

$$(\sum_{i=1}^{n}(2i-1)) = n^2$$

3. The city has 12 electric busses. The city has 40 blue buses and 20 red buses. The city employs 25 foreign drivers. 6 electric buses are blue. 8 electric buses are driven by foreign drivers. How many buses are there that have one or more of the characteristics: electric, blue, or driven by foreigners? Use the inclusion exclusion principle.

4. Let A and B be sets such that $|A| = m$ and $|B| = n$, and assume that $m \leq n$.

   (a) How many different functions $f : A \to B$ exist?

   (b) How many of these functions are injections?

   (c) If $m = n$, prove that an injection must be a bijection.

5. Let the operation $\star$ be defined for *odd integers* $\mathbb{Z}^{odd}$ such that

$$x \star y = x + y - 1$$

   Prove that $\mathbb{Z}^{odd}$ with the operation $\star$ forms a group, and explain what is the identity element of the group, and how to calculate the inverse for each element.

6. (a) What is the remainder when 4^{119} is divided by 5?

   (b) Which of the following equations have unique solutions, and why/why not? List all solutions (i.e., all equivalence classes, obviously, as there are infinite many solutions when a solution exists)

      i. $4x \equiv 7 \bmod 12$

      ii. $4x \equiv 7 \bmod 11$

      iii. $2x \equiv 6 \bmod 12$

APPENDIX: Some helpful formulas and definitions

Definition 1. If a relation is reflexive, symmetric, and transitive, we call it an *equivalence relation* or simply an *equivalence.*

Definition 2. A *group* is a pair $(G, \bullet)$ where G is a set $\bullet$ is an operation on G with the following properties

1. Associativity: $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ for a, b, and $c \in G$.
2. Identity element: There exists an element $e \in G$ such that $e \bullet a = a \bullet e = a$ for every $a \in G$.
3. Inverse: for every $a \in G$ there is an element $a^{-1} \in G$ such that $a \bullet a^{-1} = a^{-1} \bullet a = e$.

Definition 3. Let $n \in \mathbb{Z}_{+}$. If for two numbers $a, b \in \mathbb{Z}$ we have $n \mid (a - b)$, we say that a is *congruent* with b *modulo* n and we denote $a \equiv b \pmod{n}$ or $a \equiv_n b$.