SGN-11007 Introduction to Signal Processing,
Final Exam, 16.10.2018,
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- Own calculators can be used in the exam.
- You may take the examination paper with you.

1. (a) Analog signal consists of a single sine wave with the frequency 1500 Hz. The signal is sampled at intervals of $T = \frac{1}{2500}$ s.
   i. What is the Nyquist frequency? (1p)
   ii. Does aliasing happen? (1p)
   iii. What is the observed frequency of the sine wave in the digital signal after the sampling? (1p)

   (b) The pole-zero plot of a filter is in Figure 1 (left), and we know that its amplitude response $|H(e^{i\omega})| \in [0, 1]$. Sketch the amplitude response of the filter as accurately as it is possible with the information provided. (3p)

2. (a) Draw the block diagram of the system that performs sampling rate conversion from 20 kHz to
   i. 5 kHz, (1p)
   ii. 15 kHz, (1p)
   iii. 40 kHz. (1p)

   (b) Calculate the DFT of the sequence $x(n) = (5, -1, 3, 2, 0, -1, 2, 2)$ using the FFT algorithm. You can skip part of the calculations by utilizing this information: the DFT of the sequence $(5, 3, 0, 2)$ is $(10, 5-i, 0, 5+i)$ and the DFT of $(-1, 2, -1, 2)$ is $(2, 0, -6, 0)$. (3p)

3. Design using the window design method a filter (i.e. find out its impulse response) satisfying the following requirements:

Stopband	[12 kHz, 16 kHz]
Passband	[0 kHz, 9.5 kHz]
Passband ripple	0.04 dB
Minimum stopband attenuation	45 dB
Sampling frequency	32 kHz

   Use the tables below. (6p)

4. (a) The transfer function of a causal LTI system is:

$$H(z) = \frac{0.5 - z^{-1} + 1.5z^{-2}}{1 - (az)^{-1}},$$

   where the constant $a \in \mathbb{R}$ and $a \neq 0$. Determine the values of the constant a making the system stable. (3p)

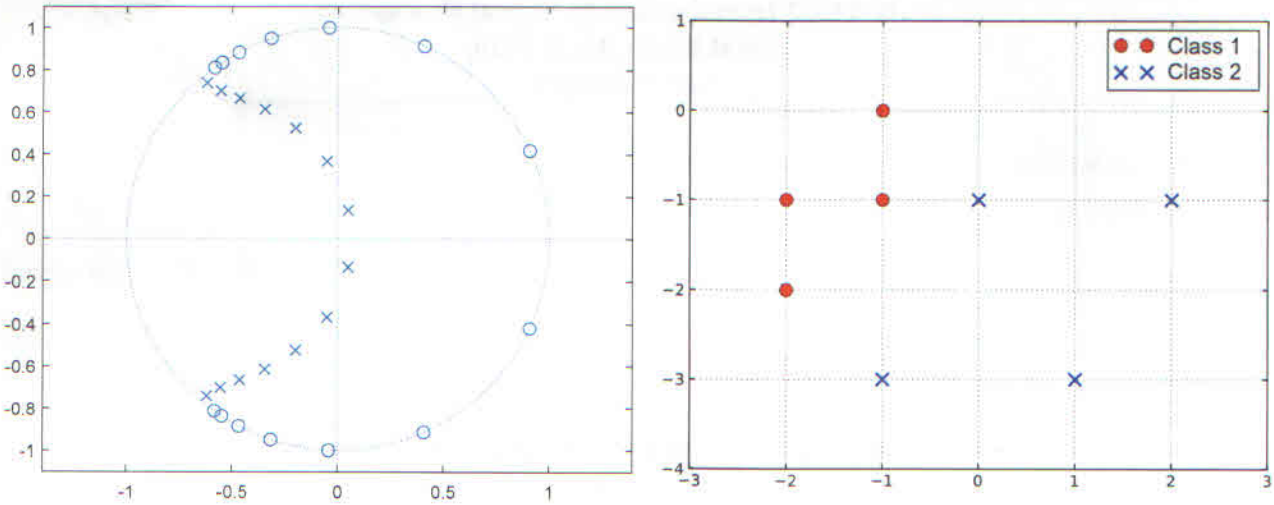

Figure 1: Left: Pole-zero plot of Task 1 (b). Right: Material of Task 5.

(b) The unit step response[1] of a system is the following

$$z(n) = \begin{cases} 0, & \text{when } n < 0 \\ 1, & \text{when } n = 0 \\ 3, & \text{when } n = 1 \\ 4, & \text{when } n \geq 2. \end{cases}$$

What is the impulse response of the system? (3p)

5. (a) Figure 1 (right) shows training data that has two classes. Copy the figure and draw the decision boundary using 1-NN classifier for the area shown in the figure. (3p)

   (b) Explain how you obtained the decision boundary in (a). As examples, give two different points on the boundary and justify why they are on the boundary. (3p)

Tables

Ideal filter type	Impulse response when	
	$n \neq 0$	$n = 0$
Low-pass	$2f_c \text{sinc}(n \cdot 2\pi f_c)$	$2f_c$
High-pass	$-2f_c \text{sinc}(n \cdot 2\pi f_c)$	$1 - 2f_c$
Band-pass	$2f_2 \text{sinc}(n \cdot 2\pi f_2) - 2f_1 \text{sinc}(n \cdot 2\pi f_1)$	$2(f_2 - f_1)$
Band-stop	$2f_1 \text{sinc}(n \cdot 2\pi f_1) - 2f_2 \text{sinc}(n \cdot 2\pi f_2)$	$1 - 2(f_2 - f_1)$

[1] Response to the unit step function $u(n) = \begin{cases} 1, & \text{when } n \geq 0, \\ 0, & \text{when } n < 0. \end{cases}$

Name of the window function	Transition bandwidth (normalized)	Passband ripple (dB)	Minimum stopband attenuation (dB)	Window expression $w(n)$, when $\|n\| \leq (N-1)/2$
Rectangular	0.9/N	0.7416	21	1
Bartlett	3.05/N	0.4752	25	$1 - \frac{2\|n\|}{N-1}$
Hanning	3.1/N	0.0546	44	$0.5 + 0.5\cos\left(\frac{2\pi n}{N}\right)$
Hamming	3.3/N	0.0194	53	$0.54 + 0.46\cos\left(\frac{2\pi n}{N}\right)$
Blackman	5.5/N	0.0017	74	$0.42 + 0.5\cos\left(\frac{2\pi n}{N}\right) + 0.08\cos\left(\frac{4\pi n}{N}\right)$

Equations

$$ax^2 + bx + c = 0 \Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\begin{cases} X(n) = X_0(n) + w_N^{-n} X_1(n), & \text{when } n = 0, 1, 2, \ldots, N/2 - 1 \\ X(n) = X_0(n - N/2) + w_N^{-n} X_1(n - N/2), & \text{when } n = N/2, N/2 + 1, \ldots, N - 1 \end{cases}$$

Some Wikipedia pages that might be useful

Suppose two classes of observations have means $\vec{\mu}_0, \vec{\mu}_1$ and covariances Σ_0, Σ_1. Then the linear combination of features $\vec{w} \cdot \vec{x}$ will have means $\vec{w} \cdot \vec{\mu}_i$ and variances $\vec{w}^T \Sigma_i \vec{w}$ for $i = 0, 1$. Fisher defined the separation between these two distributions to be the ratio of the variance between the classes to the variance within the classes:

$$S = \frac{\sigma^2_{\text{between}}}{\sigma^2_{\text{within}}} = \frac{(\vec{w} \cdot \vec{\mu}_1 - \vec{w} \cdot \vec{\mu}_0)^2}{\vec{w}^T \Sigma_1 \vec{w} + \vec{w}^T \Sigma_0 \vec{w}} = \frac{(\vec{w} \cdot (\vec{\mu}_1 - \vec{\mu}_0))^2}{\vec{w}^T (\Sigma_0 + \Sigma_1) \vec{w}}$$

This measure is, in some sense, a measure of the signal-to-noise ratio for the class labelling. It can be shown that the maximum separation occurs when

$$\vec{w} \propto (\Sigma_0 + \Sigma_1)^{-1} (\vec{\mu}_1 - \vec{\mu}_0)$$

When the assumptions of LDA are satisfied, the above equation is equivalent to LDA.

Be sure to note that the vector $\vec{w}$ is the normal to the discriminant hyperplane. As an example, in a two dimensional problem, the line that best divides the two groups is perpendicular to $\vec{w}$.

Generally, the data points to be discriminated are projected onto $\vec{w}$; then the threshold that best separates the data is chosen from analysis of the one-dimensional distribution. There is no general rule for the threshold. However, if projections of points from both classes exhibit approximately the same distributions, a good choice would be the hyperplane between projections of the two means, $\vec{w} \cdot \vec{\mu}_0$ and $\vec{w} \cdot \vec{\mu}_1$. In this case the parameter c in threshold condition $\vec{w} \cdot \vec{x} > c$ can be found explicitly:

$$c = \vec{w} \cdot \frac{1}{2}(\vec{\mu}_0 + \vec{\mu}_1) = \frac{1}{2}\vec{\mu}_1^T \Sigma_1^{-1} \vec{\mu}_1 - \frac{1}{2}\vec{\mu}_0^T \Sigma_0^{-1} \vec{\mu}_0.$$

A more condensed form of the difference equation is:

$$y[n] = \frac{1}{a_0}\left(\sum_{i=0}^{P} b_i x[n-i] - \sum_{j=1}^{Q} a_j y[n-j]\right)$$

which, when rearranged, becomes:

$$\sum_{j=0}^{Q} a_j y[n-j] = \sum_{i=0}^{P} b_i x[n-i]$$

To find the transfer function of the filter, we first take the Z-transform of each side of the above equation, where we use the time-shift property to obtain:

$$\sum_{j=0}^{Q} a_j z^{-j} Y(z) = \sum_{i=0}^{P} b_i z^{-i} X(z)$$

We define the transfer function to be:

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{\sum_{i=0}^{P} b_i z^{-i}}{\sum_{j=0}^{Q} a_j z^{-j}} \end{aligned}$$

Considering that in most IIR filter designs coefficient a_0 is 1, the IIR filter transfer function takes the more traditional form:

$$H(z) = \frac{\sum_{i=0}^{P} b_i z^{-i}}{1 + \sum_{j=1}^{Q} a_j z^{-j}}$$

Inversion of 2 × 2 matrices [edit]

The *cofactor equation* listed above yields the following result for 2 × 2 matrices. Inversion of these matrices can be done as follows:[6]

$$\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Techniques [edit]

Conceptual approaches to sample-rate conversion include: converting to an analog continuous signal, then re-sampling at the new rate, or calculating the values of the new samples directly from the old samples. The latter approach is more satisfactory, since it introduces less noise and distortion.[3] Two possible implementation methods are as follows:

1. If the ratio of the two sample rates is (or can be approximated by)[nb 1][4] a fixed rational number *L/M*: generate an intermediate signal by inserting *L* − 1 0s between each of the original samples. Low-pass filter this signal at half of the lower of the two rates. Select every *M*-th sample from the filtered output, to obtain the result.[5]
2. Treat the samples as geometric points and create any needed new points by interpolation. Choosing an interpolation method is a trade-off between implementation complexity and conversion quality (according to application requirements). Commonly used are: ZOH (for film/video frames), cubic (for image processing) and windowed sinc function (for audio).