73114 Introduction to Functional Analysis

Examination 10.5.2001

No notes, books or calculator

Problem 1. Let X and Y be normed spaces and $A \in \mathcal{L}(X;Y)$. Show that if (x_n) is a Cauchy sequence in X, then (Ax_n) is a Cauchy sequence in Y.

Problem 2. Let X be a normed space, $A \in \mathcal{L}(X)$ and $c > 0$. Express

$$\sup_{\|x\| \le c} \|Ax\|$$

in terms of the norm of A.

Problem 3. Let X be a Hilbert space and $A \in \mathcal{L}(X)$. Show that $\mathcal{N}(A) = \mathcal{N}(A^*A)$, where $\mathcal{N}(A) = \{ x \in X \mid Ax = 0 \}$.

Problem 4. Let X and Y be normed spaces and $f : X \to Y$ continuous at $x_0 \in X$. Show that if $f(x_0) \neq 0$ then there is an open ball $B(x_0, r)$ such that $f(x) \neq 0$ for every $x \in B(x_0, r)$.

Problem 5. Let ℓ^2 be the sequence space

$$\ell^2 = \{ (x_n)_{n=1}^{\infty} \subset \mathbb{C} \mid \textstyle\sum_{n=1}^{\infty} |x_n|^2 < \infty \}.$$

Define the operator $A : \ell^2 \to \ell^2$ by

$$Ax = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots, \frac{x_n}{n}, \ldots \right), \qquad x = (x_n)_{n=1}^{\infty} \in \ell^2.$$

(a) Determine the spectrum, $\sigma(A)$, of A, and provided that A has eigenvalues, the corresponding eigenvectors.

(b) Show that A is a compact operator ($1/n \to 0$ is not acceptable as proof).